whence, bearing in mind that $f(x)$, being continuous, is the differential coefficient of its integral, we get at once

$$f(x)-f(a)=\int_a^x f'(x)\,dx.$$

Here the sign of integration refers to integration of the generalised Lebesgue type, and it will be sufficiently evident from what precedes, that this equation is only then in general true when the sign of integration is interpreted in this sense. It may be remarked that $f'(x)$ is both a *lu* and an *ul*, and is, therefore, of a very elementary type in our scheme of functions.

On the Formation of Usually Convergent Fourier Series.

By W. H. Young, Sc.D., F.R.S., Professor of Mathematics, University of Liverpool.

(Received December 19, 1912,—Read January 30, 1913.)

§ 1. Series which converge except at a set of content zero, or, using the expression very commonly adopted, series which converge usually, possess many of the properties which appertain to series which converge everywhere. It becomes, therefore, of importance to devise circumstances under which we can assert the consequence that a series converges in this manner. The subject has recently received considerable attention. So far as Fourier series are concerned no result of even an approximately final character has been obtained. It may be supposed, indeed, that the results* of Jerosch and Weyl were at first so regarded,† but, if we examine them closely in the light of the Riesz-Fischer theorem, which was known previously to the results of these authors, it becomes evident that they are merely *equivalent* to the statement that the Fourier series of a function, whose square is summable, is changed into one which converges usually, if the typical coefficients a_n and b_n are divided by the sixth root of the integer n denoting their place in the series. Now it is difficult to believe that the question of the usual convergence of a Fourier series can depend on the degree of the summability of the function with which it is associated,

* 'Math. Ann.,' vols. 66 and 67.

† The result due to Fatou that a series of Fourier converges usually if na_n and nb_n converge to zero is still more special, being of course included in Jerosch's condition. For Fatou's paper, see 'Acta Mat.,' vol. 30.

and it is still more difficult to see how precisely the sixth root of n can have anything to do with it. On the other hand Weyl's method, which itself marks an advance on that of Jerosch, does not obviously lend itself to any suitable modification which would secure a greater degree of generality in the result.

The mistake is frequently made of confusing theoretical interest with practical importance in the matter of a necessary and sufficient test. Tests which are only sufficient, but not necessary, are often much more convenient. Still more frequently it is convenient to work from first principles, and not to use any test at all. Instead of employing Weyl's necessary and sufficient condition that a series should converge usually, I have attacked the problem directly. The principles I have employed do not differ essentially from those already exposed in previous communications to this Society, but the generality and interest of the results obtained in the matter in hand seem to justify a further communication.

These results are as follows:—

The Fourier series of any function whatever, of any degree of summability, and its allied series, are both of them changed into Fourier series which converge usually, if the coefficients a_n and b_n are divided by any power, however small, of the index n denoting their place in the series, in other words, they are converted into such series by the use of the convergence factor n^{-k}, $(0 < k)$, and by the use of the convergence factor $(\log n)^{-1-k}$.

More generally, they are converted into such series by the use of the convergence factor whose numerator is unity, and whose denominator is

$$l_1(n)\, l^2_2(n) \ldots l^2_{r-1}(n)\, [l_r(n)]^{2+k},\ (0 < k),\ \textit{where } l_r(n) = \log l_{r-1}(n), \textit{ and } l_0(n) = n.$$

I have thought it sufficient to prove this latter theorem for the case in which $r = 2$. With this before him the reader will easily be able, by induction or otherwise, to carry out the proof of the general result.

It will be remarked that, unlike Weyl, I have not considered general series of orthogonal functions. The theory of these functions involves usually greater analytical difficulties, but not a greater wealth of ideas, and I have thought it best, with so many problems in the theory of Fourier series still unsolved, to confine my attention to this simpler class of series. The detailed study of the theory of Fourier series seems to me to form the best possible preparation for the larger theory.

Finally, it should be noticed that the results arrived at do not in any way strengthen the probability suggested by Lusin that the Fourier series of a function whose square is summable necessarily converges usually.

Lusin's statement that this is infinitely probable seems based on the fact that in any succession of partial summations of such a series a sub-succession can be found which usually converges to the function. This is, however, quite consistent with, for example, finite oscillation except at a set of content zero. On the other hand, the results perhaps do suggest the possibility, though scarcely the infinite probability, of this latter circumstance presenting itself for all Fourier series. If this be true, every monotone succession of constants, having zero as limit, would have the effect of the special series of constants employed in the theorems of the present communication. These latter, however, are so far from being the most general successions of the monotone type that they are the Fourier coefficients both of sine and of cosine series.

§ 2. In the mode of investigation I have adopted, the following theorem and its analogue are fundamental :—

Theorem.—*Let $f(x)$ be a summable function whose typical Fourier cosine and sine constants are a_n and b_n, and let $q_1, q_2, \ldots, q_n, \ldots$, be a monotone descending sequence of constants with zero as limit, then the series*

$$\sum_{r=1} q_r (a_r \cos rx + b_r \sin rx), \tag{1}$$

oscillates boundedly at the point x, provided a succession of constants k_n can be found such that

(i) *q_n/k_n is bounded for all values of n,*

(ii) *The series $\sum_{r=1} (q_r - q_{r+1})/k_r$ converges,*

(iii) *$\int_{-\pi}^{\pi} k_n f(x+u) \sin(n+\frac{1}{2}) u \operatorname{cosec} \frac{1}{2} u \, du$ is a bounded function of n.*

Let $S_n(x)$ denote the partial summation consisting of the first n terms of (1), and let

$$s_n = \sum_{r=1}^{r=n} q_r \cos rx. \tag{2}$$

Then, since $a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx \, dx$, and $b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin rx \, dx$, we get, regarding $f(x)$ as periodic with period 2π,

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u)\, s_n(u) \, du. \tag{3}$$

Now
$$2 s_n(u) \sin \tfrac{1}{2} u = \sin \tfrac{1}{2} u + \sum_{r=1}^{r=n} q_r [\sin(r+\tfrac{1}{2}) u - \sin(r-\tfrac{1}{2}) u]$$

$$= q_n \sin(n+\tfrac{1}{2}) u + \sum_{r=1}^{r=n-1} (q_r - q_{r+1}) \sin(r+\tfrac{1}{2}) u. \tag{4}$$

Therefore, by (3) and (4),

$$2\pi S_n(x) = q_n t_n + \sum_{r=1}^{r=n-1} (q_r - q_{r+1})\, t_r, \tag{5}$$

where $$t_r = \int_{-\pi}^{\pi} f(x+u) \sin(r+\tfrac{1}{2})u \operatorname{cosec} \tfrac{1}{2}u\, du,$$

and is accordingly, to a constant factor *près*, the n-th partial summation of the Fourier series of $f(x)$.

From (5) the truth of the theorem is at once evident, since by the hypothesis (iii) $t_r k_r$ is a bounded function of r, while the conditions (i) and (ii) are satisfied.

§ 3. The corresponding theorem for the allied series of $f(x)$ is as follows :—

Theorem.—*Let $f(x)$ be a summable function, whose typical Fourier cosine and sine constants are a_n and b_n, and let $q_1, q_2, \ldots, q_n, \ldots,$ be a monotone descending sequence of constants with zero as limit, then the series*

$$\sum_{r=1} q_r (b_r \cos rx - a_r \sin rx) \tag{1}$$

oscillates boundedly at the point x, provided a succession of constants k_n can be found, such that

(i) *q_n/k_n is bounded for all values of n,*

(ii) *The series $\sum_{r=1}(q_r - q_{r+1})/k_r$ converges,*

(iii) *$\int_{-\pi}^{\pi} k_n f(x+u)\,[1-\cos(n+\tfrac{1}{2})u] \operatorname{cosec} \tfrac{1}{2}u\, du$ is a bounded function of n.*

Let $S_n(x)$ denote the partial summation consisting of the first n terms of (1) and let

$$s_n = \sum_{r=1}^{r=n} q_r \sin rx.$$

Then, as in § 2, $$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u)\, s_n(u)\, du. \tag{2}$$

Now

$$2s_n(u) \sin \tfrac{1}{2}u = \sum_{r=1}^{r=n} q_r \{\cos(r-\tfrac{1}{2})u - \cos(r+\tfrac{1}{2})u\}$$

$$= q_n\{1-\cos(n+\tfrac{1}{2})u\} - q_1(1-\cos\tfrac{1}{2}u) + \sum_{r=1}^{r=n-1} (q_r - q_{r+1})\{1-\cos(r+\tfrac{1}{2})u\}. \tag{3}$$

Therefore, by (2) and (3)

$$2\pi S_n(x) = q_n \tau_n - q_1 \tau_1 + \sum_{r=1}^{r=n-1} (q_r - q_{r+1})\, \tau_r, \tag{4}$$

where $$\tau_r = \int_{-\pi}^{\pi} f(x+u)\{1-\cos(r+\tfrac{1}{2})u\} \operatorname{cosec} \tfrac{1}{2}u\, du, \tag{5}$$

and is accordingly, to a constant factor *près*, the nth partial summation of the allied series of the Fourier series of $f(x)$.

From (4) the truth of the theorem is at once evident, since, by the hypothesis (iii), $\tau_r k_r$ is a bounded function of r, while q_n/k_n and $\sum_{r=1}^{r=n-1}(q_r-q_{r+1})/k_r$ are bounded functions of n.

§ 4. In applying these theorems we shall require some elementary results in the convergence of series.

Lemma 1.—*If* $$q_n = 1/[l_r(n)]^q \qquad (0<q),$$ *where r has any positive integral value, including zero, and*
$$l_0(n) = n, l_1(n) = \log n, \dots l_r(n) = \log l_{r-1}(n),$$
then
$$q_n - q_{n+1} \leq q/\prod_{s=0}^{s=r-1} l_s(n)[l_r(n)]^{1+q}.$$

Writing $$f(x) = 1/[l_r(x)]^q,$$
we have $$f'(x) = -q/\prod_{s=0}^{s=r-1} l_s(x) \,.\, [l_r(x)]^{1+q}.$$

But $$q_n - q_{n+1} = -f'(x_1) \qquad (n<x_1<n+1),$$
$$\leq -f'(n),$$
which proves the lemma.

Cor.—*If $k<q$, and k_n has the same form as q_n, with k instead of q, the series whose general term is $(q_n-q_{n+1})/k_n$ converges.*

Lemma 2.—*If* $$q_n = 1/l_1(n)[l_2(n)]^{2+q} \qquad (0<q),$$
and $$k_n = 1/l_1(n)[l_2(n)]^{1+k} \qquad (k<q),$$
the series $\sum_{n=2}(q_n-q_{n+1})/k_n$ *is convergent.*

Writing $$f(x) = 1/l_1(x)[l_2(x)]^{2+q},$$
we have $$f'(x) = -f(x)\left(\frac{1}{xl_1(x)}+\frac{2+q}{xl_1(x)\,l_2(x)}\right).$$

But $$q_n - q_{n+1} = -f'(x_1) \qquad (n<x_1<n+1),$$
whence
$$(q_n-q_{n+1})/k_n \leq -f'(n)/k_n \leq \frac{1}{l_2(n)^{1+q-k}}\left(\frac{1}{nl_1(n)}+\frac{2+q}{nl_1(n)\,l_2(n)}\right).$$

Thus the series whose convergence is under discussion is not greater than the sum of two series both known to be convergent, namely,
$$\sum_{n=2} 1/nl_1(n)[l_2(n)]^{1+q-k} \quad \text{and} \quad (2+q)\sum_{n=2} 1/nl_1(n)[l_2(n)]^{2+q-k},$$
This proves the lemma.

It should be remarked that if the index of the power of $l_2(n)$ in the

expression for q_n be $-1-q$ instead of $-2-q$, the series in question may be easily seen to be divergent.

Lemma 3.—*If*

$$q_n = 1/l_1(n)[l_2(n)]^2\,[l_3(n)]^2\ldots[l_r(n)]^{2+q},$$

and

$$k_n = 1/l_1(n)\,l_2(n)\,l_3(n)\ldots[l_r(n)]^{1+q},$$

the series $\Sigma\,(q_n - q_{n+1})/k_n$ *is convergent.*

Writing

$$f(x) = 1/l_1(n)[l_r(n)]^{2+q}\prod_{s=2}^{s=r-1}[l_s(n)]^2,$$

we have

$$f'(x) = -f(x)\left\{\frac{1}{xl_1(x)} + \frac{2+q}{xl_1(x)l_2(x)\ldots l_r(x)} + \sum_{s=2}^{s=r-1}\frac{2}{xl_1(x)\,l_2(x)\ldots l_r(x)}\right\}.$$

Hence, as in preceding proof, our series is the sum of r series known to be convergent, which proves the lemma.

§ 5. In this and the next articles we shall prove certain properties as to the order of infinity of the partial summations of a Fourier series, and of its allied series, except at a set of content zero.

Theorem.—*If* $f(x)$ *be any summable function of* x, *and* $k_n = n^{-k}$, $(0<k)$, *the integrals*

$$\int_{-\pi}^{\pi} k_n f(x+u)\sin(n+\tfrac{1}{2})u\operatorname{cosec}\tfrac{1}{2}u\,du \quad \textit{and} \quad \int_{-\pi}^{\pi} k_n f(x+u)[1-\cos(n+\tfrac{1}{2})u]\operatorname{cosec}\tfrac{1}{2}u\,du$$

are, for each value of x *not belonging to a certain set of zero content, bounded functions of* n.

In fact the former integral may be written

$$\int_{-\pi}^{\pi} f(x+u)\,(\operatorname{cosec}\tfrac{1}{2}u)^{1-k}\,[n^{-k}\sin(n+\tfrac{1}{2})\,u\,(\operatorname{cosec}\tfrac{1}{2}u)^k]\,du,$$

the absolute value of which is therefore

$$\leq \int_{-\pi}^{\pi} |f(x+u)|\,|\operatorname{cosec}\tfrac{1}{2}u|^{1-k}C_k\,du,$$

where C_k is a constant depending only on k.*

Here we have tacitly supposed that the integral just written down exists, which will certainly be the case, except for a set of values of x of zero content, in virtue of a theorem that I have proved elsewhere,† since $f(n)$ and $(\operatorname{cosec}\tfrac{1}{2}u)^{1-k}$ are summable functions of u.

In precisely the same way the second integral is proved to be less in

* See W. H. Young, "The Convergence of Certain Series Involving the Fourier Constants of a Function," 'Roy. Soc. Proc.,' 1912, A, vol. 87, p. 221.

† "Sur la généralisation du théorème de Parseval," 'Comptes Rendus,' 1912, vol. 135, p. 30 ; séance du 1er juillet.

absolute value than a quantity which is finite except at a set of content zero, using the boundedness of $\dfrac{1-\cos t}{t^k}$ instead of that of $\dfrac{\sin t}{t^k}$.

§ 6. We have given a separate statement and proof of the theorem of the last article because of its greater simplicity. We can, however, still further approximate to the order of infinity of the successions of our summations except at a set of content zero.

Theorem.—*If $f(x)$ be a summable function of x and*

$$k_n = (\log n)^{-1-k} \qquad (0 < k),$$

the integrals

$$\int_{-\pi}^{\pi} k_n f(x+u)\sin(n+\tfrac{1}{2})u \operatorname{cosec}\tfrac{1}{2}u\,du \quad \textit{and} \quad \int_{-\pi}^{\pi} k_n f(x+u)[1-\cos(n+\tfrac{1}{2})u]\operatorname{cosec}\tfrac{1}{2}u\,du$$

are, for each value of x not belonging to a certain set of content zero, bounded functions of n.

We have

$$\begin{aligned} |\sin mu|/(\log m)^{1+k} &= |\sin mu\,(\log u)^{-1-k}|\,\{(\log mu - \log m)/\log m\}^{1+k} \\ &\leq |\log u|^{-1-k}\,(|\sin mu|\,|\log mu|^{1+k}(\log m)^{-1-k}+1), \end{aligned}$$

and therefore $\qquad \leq C_k\,|\log u|^{-1-k}$,

since $\sin t\,|\log t|^{1+k}$ is bounded in any closed neighbourhood of the origin, and $\log mu/\log m$ remains bounded as m increases indefinitely.

Hence since

$$\operatorname{cosec}\tfrac{1}{2}u\,|\log u|^{-1-k}$$

is summable in an interval enclosing the origin not containing the point $u = 1$, the result stated with respect to the first integral is true, by reasoning similar to that employed in § 2, bearing in mind that it is clearly sufficient to prove the property in question for an interval of integration not including the point $u = 1$.

Precisely similar reasoning proves the statement with respect to the second integral, using the fact that $(1-\cos t)\,|\log t|^{1+k}$ is bounded.

§ 7. By a slightly more complicated process we arrive at the following result:—

Theorem.—*If $f(x)$ be a summable function of x, and*

$$k_n = (\log n)^{-1}(\log\log n)^{-1-k}, \qquad (0 < k),$$

the integrals

$$\int_{-\pi}^{\pi} k_n f(x+u)\sin(n+\tfrac{1}{2})u \operatorname{cosec}\tfrac{1}{2}u\,du$$

and

$$\int_{-\pi}^{\pi} k_n f(x+u)[1-\cos(n+\tfrac{1}{2})]u \operatorname{cosec}\tfrac{1}{2}u\,du$$

are, for each value of x not belonging to a certain set of content zero, bounded functions of n.

We have

$$\left|\sin mu\right|\left|\log|u|\right|\left|\log|\log u|\right|^{1+k}\Big/\log m\,(\log\log m)^{1+k}$$
$$=\left|\sin mu\right|\left|\frac{\log|mu|}{\log m}-1\right|\left|\frac{\log\left|\log|mu|-\log m\right|}{\log\log m}\right|^{1+k}.$$

For convenience of printing the quantities which now occur are to be supposed all taken in absolute magnitude. Moreover, as in § 6, we may confine our attention to a conveniently small interval containing the origin, and not containing either the point unity, nor $1/e$, and we need only consider the positive part of this interval. We have then

$$\sin mu\log u\,(\log\log u)^{1+k}/\log m\,(\log\log m)^{1+k}$$
$$=\sin mu\left(\frac{\log mu}{\log m}-1\right)\left(\frac{\log(\log mu-\log m)}{\log\log m}\right)^{1+k}$$
$$\leq \sin mu\left(\frac{\log mu}{\log m}-1\right)\left(\frac{\log\log mu}{\log\log m}+1\right)^{1+k}$$
$$\leq \sin mu\left(\frac{\log mu}{\log m}-1\right)2^k\left[\left(\frac{\log\log mu}{\log\log m}\right)^{1+k}+1\right].$$

Multiplying out, we get the sum of four terms, each of which is bounded, namely

$$\frac{\sin mu\log mu\,(\log\log mu)^{1+k}}{\log m\,(\log\log m)},\quad \frac{\sin mu\log mu}{\log m},\quad \sin mu\left(\frac{\log\log mu}{\log\log m}\right)^{1+k},$$

and $\sin mu$,

each multiplied by 2^k.

Hence, since $$1/\sin\tfrac{1}{2}u\log u\,(\log\log u)^{1+k}$$

is summable, the required result follows as before, for the first integral.

In a precisely similar way it follows for the second integral.

§ 8. The general theorem that I have been establishing step by step with regard to the order of infinity of the successions of our summations is now evident. It is as follows:—

Theorem.—*If $f(x)$ is a summable function of x, the n-th partial summation of the Fourier series of $f(x)$, and the corresponding summation connected with its allied series have an order of infinity which, for every r and every k, is for every x not belonging to a certain set of content zero, less than that of*

$$l_1(n)\,l_2(n)\ldots[l_r(n)]^{1+k}.$$

The reader may be left to complete the proof by induction, or otherwise.

§ 9. We are now able to prove the main theorems which form the subject of the paper. We have only to take the fundamental theorems of §§ 2 and 3

and apply to them the suitable lemma and auxiliary theorem. These theorems enable us in the first instance only to state that the series in question oscillate boundedly except for values of x forming a set of content zero. As, however, the quantity q is at our disposal, we may perform the process of dividing by n^q in two stages, first by $n^{q'}$, and then by $n^{q-q'}$, where $0 < q' < q$. The latter process turns the finite oscillation into convergence, by Abel's Lemma. Similar remarks apply in the case of the other convergence factors.

Theorem 1.—*If $f(x)$ is a summable function, whose typical Fourier cosine and sine constants are a_n and b_n, then the series whose general terms are respectively*

$$n^{-q}(a_n \cos nx + b_n \sin nx) \qquad (0 < q)$$

and

$$n^{-q}(b_n \cos nx - a_n \sin nx)$$

both converge usually.

For, since
$$n^{-q}/n^{-k} = n^{-(q-k)} \leq 1 \qquad (k < q),$$
the condition (i) of the theorem of § 2 is satisfied.

Also by the Lemma of 1 § 4 the condition (ii) is satisfied.

Finally the condition (iii) is satisfied by § 5. Hence by the theorem of § 2 and Abel's Lemma the former series converges usually.

Similarly the latter series converges usually, using the theorem of § 3 instead of that of § 2.

Theorem 2.—*Under the same circumstances as in the preceding theorem, the series*

$$\sum_{n=2} (\log n)^{-1-q}(a_n \cos nx + b_n \sin nx) \qquad (0 < q),$$

and

$$\sum_{n=2} (\log n)^{-1-q}(b_n \cos nx - a_n \sin nx)$$

converge usually.

For since
$$(\log n)^{-1-q}/(\log n)^{-1-k} = (\log n)^{-(q-k)} \leq (\log 2)^{-(q-k)} \qquad (k < q),$$
the condition (i) of §§ 2 and 3 is satisfied.

The condition (ii) is satisfied in virtue of the Lemma 1 of § 4.

Finally the condition (iii) is satisfied by § 6.

Hence by §§ 2 and 3 and Abel's Lemma the theorem is true.

Theorem 3.—*Under the same circumstances, the series*

$$\sum_{n=3} (\log n)^{-1}(\log\log n)^{-2-q}(a_n \cos nx + b_n \sin nx),$$

$$\sum_{n=3} (\log n)^{-1}(\log\log n)^{-2-q}(b_n \cos nx - a_n \sin nx),$$

converge usually.

Here we take $$q_n = 1/l_1(n)\,[l_2(n)]^{2+q},$$
and $$k_n = 1/l_1(n)\,[l_2(n)]^{1+k}, \qquad (k < q).$$
We then have $$q_n/k_n = 1/[l_2(n)]^{1+q-k},$$
showing that the condition (i) of §§ 2 and 3 is satisfied.

The condition (ii) is satisfied in virtue of the Lemma 2 of § 4.

Finally the condition (iii) is satisfied by § 7.

Hence by §§ 2 and 3 the theorem is true, using once more Abel's Lemma.

Theorem 4.—*Under the same circumstances the series*

$$\Sigma\, q_n(a_n \cos nx + b_n \sin nx) \quad \textit{and} \quad \Sigma\, q_n(b_n \cos nx - a_n \sin nx),$$

where $$q_n = 1/l_1(n)\,[l_2(n)]^2 \ldots [l_r(n)]^{2+q} \qquad (0 < q),$$

converge usually.

Here we take
$$k_n = 1/l_1(n)\, l_2(n) \ldots [l_r(n)]^{1+k} \qquad (k < q).$$

We then have
$$q_n/k_n = 1/l_2(n)\, l_3(n) \ldots [l_r(n)]^{1+q-k},$$
showing that the condition (i) of §§ 2 and 3 is satisfied.

The condition (ii) is satisfied in virtue of Lemma 3 of § 4.

Finally the condition (iii) is satisfied by § 8.

Hence by §§ 2 and 3 the theorem is true, using once more Abel's Lemma.

§ 10.* If, instead of the theorem that $\int f(x+t)\,g(t)\,dt$ exists as a function of x *almost* everywhere, when f and g are summable functions, we employ the connected result that this function exists *everywhere* and is continuous, when the summabilities of f and g are suitably connected, we obtain in a similar manner information as to the order of infinity *everywhere* of the partial summations of the Fourier series of the various types of functions. In this way not only are results such as those exposed in my papers "On the Convergence of Certain Series, etc.,"† and "On the Fourier Series of Bounded Functions,"‡ confirmed, but further new ones are obtained.

§ 11. On the other hand, Mr. G. H. Hardy points out that it is possible to slightly extend the results of this paper as regards Fourier series by utilising a theorem of Lebesgue's instead of the above-mentioned theorem. Indeed, we are thus enabled, in the case both of the Fourier series and its allied series, to replace the two's by unities in the general convergence factor. Moreover, in the case of the Fourier series—as distinct, be it said, from the allied series—we can, as he remarks, go still farther and obtain a further slight, but, as it

* §§ 10–12 have been added during the passage through press, Feb. 8, 1913.

† *Loc. cit. supra.*

‡ 'Lond. Math. Soc. Proc.,' 1912, Ser. 2, vol. 12.

appears to me, important extension, by utilising an additional artifice already employed by Messrs. Hardy and Littlewood.

§ 12. We may similarly, when desirable, instead of the connected result referred to in § 10, use a property of an integral of a function of given kind of summability, such as follows from considerations exposed in my paper on "Summable Functions and their Fourier Series."*

On a Cassegrain Reflector with Corrected Field.

By R. A. Sampson, F.R.S.

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(Abstract.)

The purpose of this memoir is to discover an optical appliance which shall correct in a practical manner the faults in the field of a Cassegrain reflector, while leaving unimpaired its achromatism and the characteristic features of its design, which gives a focal length much greater than the length of the instrument, combined with a convenient position of the observer. The question touches an investigation by Schwarzschild† as to what can be done with two curved mirrors the figures of which are not necessarily spherical. With these he corrects spherical aberration and coma, but in order to secure a flat field he is led to a construction in which the second mirror, which is between the great mirror and its principal focus, is concave, and therefore shortens the effective focal length, in place of increasing it. The deformations from spherical figures are also so great, especially for the great mirror, as to leave it doubtful whether the construction discussed could ever be the model for practicable instruments. If we keep to the Cassegrain form, spherical aberration and coma may equally be corrected by deformations of the mirrors which, though large, are less extreme, but there remains a pronounced curvature of the field. For this reason I am led, in the present memoir, to consider more complicated systems produced by the interposition of systems of lenses. Achromatism can be preserved completely for a single focus if there are three lenses of focal length determined when their position are given, and if all are made of the same glass. One of these lenses, which I

* 'Roy. Soc. Proc.,' 1912.

† 'K. Gesell. d. Wissenschaften zu Göttingen, Math.-Phys.-Classe,' Neue Folge, 1905, vol. 4.